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DIFFERENTIAL CONSTITUTIVE EQUATIONS OF INCOMPRESSIBLE MEDIA WITH FINITE DEFORMATIONS

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A method is proposed for constructing a system of constitutive equations of an incompressible medium with nonlinear dissipative properties with finite deformations. A scheme of the mechanical behavior of a material is used, in which the points are connected by horizontally aligned elastic, viscous, plastic, and transmission elements. The properties of each element of the scheme are described with the use of known equations of the nonlinear elasticity theory, the theory of nonlinear viscous fluids, and the theory of plastic flow of the material under conditions of finite deformations of the medium. The system of constitutive equations is closed by equations that express the relation between the deformation rate tensor of the material and the deformation rate tensor of the plastic element. Transmission elements are used to take into account a significant difference between macroscopic deformations of the material and deformations of elements of the medium at the structural level.

Key words: *constitutive equations, finite deformations, elasticity, plasticity, viscosity, incompressible medium, mechanical properties.*

Introduction. The viscoelastic behavior and viscous-flow properties of materials with finite deformations are simulated with the use of models of integral and differential types. In models of the integral type, the history of medium deformation is taken into account by means of integral equations. It is difficult to extend these equations to more complicated situations (for instance, for modeling processes of mass transfer in a dissipative medium, which play the governing role in the technology of production of polymer materials). If damageability and thixotropy of elastomers are taken into account, integral models become too complicated for application in practice. It seems more promising to construct differential phenomenological models. These models are simple and convenient for calculations and identification of constants on the basis of available experimental data and can describe real mechanical properties of materials fairly accurately. Apparently, this is the reason for the recently increased interest to differential models of continuous media with finite deformations.

Differential models usually describe the rheological properties of materials with the use of tensor internal variables, which are given the physical meaning of stresses [1–5] or strains [6–12]. It is convenient to construct mathematical models of this type with the use of schemes of the mechanical behavior of the medium. Available publications, however, do not provide a general theory of constructing differential-type models with arbitrary connections of elements for media with finite deformations. Such models cannot be obtained by simple generalization of models for media with small deformations, because the total measure of deformation cannot be presented as a sum of the measures of deformation of individual elements of the scheme. Equations of the evolution of internal variables should involve objective derivatives whose choice has to be properly justified.

Rheological models, which have simple interpretations in the form of the Maxwell, Kelvin–Voigt, Pointing–Thompson, and other schemes, were formulated in [13]. The thermodynamic validity of simple constitutive equations was discussed in [14, 15]. We believe that Palmov's idea are of the greatest interest. In the present paper, we use a similar approach based on additive decomposition of the deformation rate tensor of the medium.

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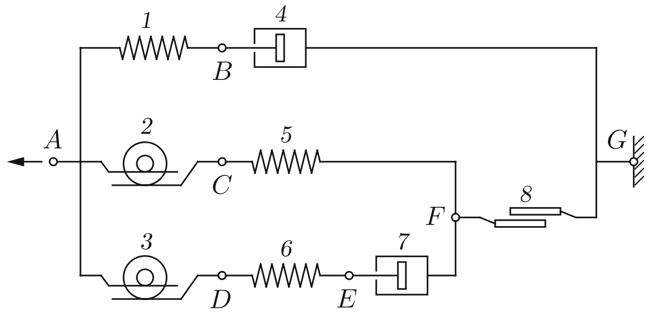

Fig. 1. One possible scheme of the mechanical behavior of the material: elastic elements (1, 5, and 6), transmission elements (2 and 3), viscous elements (4 and 7), and a plastic element (8).

A simple method for deriving the constitutive equations is proposed below. Advantages of this method are the existence of the physical meaning of all mathematical expressions used, the convenience of constructing the mathematical model of the medium, and the absence of objective derivatives of stress tensors in the constitutive equations. Numerous methods of connecting the elements in the scheme offer additional possibilities for the development of a model that provides a fairly accurate description of the medium behavior with the minimum number of internal variables.

1. Scheme of the Mechanical Behavior of the Medium and Construction of the System of Constitutive Equations. To construct the system of constitutive equations of complex media, we propose to use the scheme of the mechanical behavior of the material. An example of such a scheme is illustrated in Fig. 1. Some rules that have to be satisfied in constructing the mathematical model of the material are given below:

— the scheme of the mechanical behavior of the medium should consist of points connected by elastic, viscous, plastic, and transmission elements;

— all elements on the scheme should be aligned horizontally; therefore, we can speak about the left and right points of the corresponding elements;

— the elements are connected to the left or right point in the scheme and cannot be connected to the top or bottom point;

— for each point of the scheme, there is a corresponding deformation rate tensor of this point, which plays the role of a tensor parameter necessary for constructing the mathematical model;

— the Cauchy stress tensor and the deformation rate tensors are assigned to the elastic, viscous, and plastic elements of the scheme;

— for each transmission element, the Cauchy stress tensors for the left and right points are used.

A specific feature of the approach considered is the fact that it does not involve the notion of deformation gradients for internal points in the scheme of the mechanical behavior of the medium. In using this approach, therefore, we cannot speak about multiplicative decomposition of the deformation gradient of the material into a product of deformation gradients of the scheme elements. A specific feature of this approach is the use of transmission elements.

In the example considered (see Fig. 1), the scheme of the mechanical behavior of the medium consists of points A, B, C, D, E, F, and G connected by elastic elements 1, 5, and 6, transmission elements 2 and 3, viscous elements 4 and 7, and a plastic element 8. Deformation rate tensors with the corresponding subscripts are put into correspondence to the points in the scheme: D_A, D_B, D_C, D_D, D_E, D_F, and D_G. These deformation rate tensors are used to calculate the deformation rate tensors of elastic, viscous, and plastic elements. The left A and right G points of the scheme of the mechanical behavior of the medium play an important role. The algorithm of obtaining the system of constitutive equations is based on using statements given below.

Statement 1. The deformation rate tensor of the left point of the scheme coincides with the deformation rate tensor of the medium D, and the deformation rate tensor of the right point of the scheme is equal to zero.

Statement 2. The deformation rate tensor of the elastic, viscous, and plastic elements is calculated as the difference between the deformation rate tensors of the left and right points of these elements.

Statement 3. The trace of any deformation rate tensor used in the model is equal to zero.

In the example considered, these statements mean that the equalities

$$D_A = D, \qquad D_G = 0 \tag{1}$$

should be used in constructing the mathematical model for the left and right points of the scheme,

$$D_1 = D_A - D_B, \qquad D_5 = D_C - D_F, \qquad D_6 = D_D - D_E \tag{2}$$

should be used for the elastic elements, and

$$D_4 = D_B - D_G, \qquad D_7 = D_E - D_F, \qquad D_8 = D_F - D_G \tag{3}$$

should be used for the viscous and plastic elements. The requirements of zero traces of the deformation rate tensors

$$\operatorname{tr} D_1 = \ldots = \operatorname{tr} D_8 = \operatorname{tr} D_A = \ldots = \operatorname{tr} D_G = 0$$

is caused by the fact that the constitutive equations are derived for the model of an incompressible medium.

Statement 4. The transmission element is used in the model for changing the deformation rate and cannot be responsible for energy loss or production. The power of the Cauchy stress tensor T_k^l on the deformation rate tensor D_k^l at the left point of the transmission element with the number k equals the power of the Cauchy stress tensor T_k^r on the deformation rate tensor D_k^r at the right point of this element:

$$T_k^l \cdot D_k^l = T_k^r \cdot D_k^r \tag{4}$$

(the dot means scalar multiplication of nine-dimensional vectors in the nine-dimensional vector space formed by the set of second-rank tensors).

In accordance with condition (4), the following equalities have to be satisfied in the example considered:

$$T_2^l \cdot D_A = T_2^r \cdot D_C, \qquad T_3^l \cdot D_A = T_3^r \cdot D_D. \tag{5}$$

The model involves the Cauchy stress tensors of the elastic elements T_1, T_5, and T_6, viscous elements T_4 and T_7, and plastic element T_8, as well as the Cauchy stress tensors of the left points (T_2^l and T_3^l) and right points (T_2^r and T_3^r) of the transmission elements.

The next step in constructing the system of constitutive equations is the formulation of conditions of compatibility of the Cauchy stress tensors. The following statements are used for this purpose.

Statement 5. The Cauchy stress tensor of the medium T is equal to the sum of the Cauchy stress tensors of the elastic, viscous, and plastic elements and of the left points of the transmission elements connected to the left point of the scheme.

Statement 6. The sum of the Cauchy stress tensor of the elastic, viscous, and plastic elements and the right points of the transmission elements connected on the left to an arbitrary internal point of the scheme is equal to the sum of the Cauchy stress tensors of the elastic, viscous, and plastic elements and the left points of the transmission elements connected on the right to this point of the scheme.

Using the above-formulated statements for this model (see Fig. 1), we obtain the following dependences. The Cauchy stress tensor of the material T is equal to the sum of the Cauchy stress tensors of the elastic element T_1 and the Cauchy stress tensors of the left points of the transmission elements T_2^l and T_3^l:

$$T = T_1 + T_2^l + T_3^l. \tag{6}$$

The following equalities are valid for the internal points B, C, D, and E:

$$T_1 = T_4, \qquad T_2^r = T_5, \qquad T_3^r = T_6, \qquad T_6 = T_7. \tag{7}$$

The point F obeys the condition

$$T_5 + T_7 = T_8. \tag{8}$$

2. Mechanical Properties of the Elastic Elements. To describe the mechanical behavior of the elements in the scheme, we use the known formulas of mechanics. Thus, the mass density of the free energy can be used to calculate the Cauchy stress tensors in the elastic elements.

Statement 7. The mass density of the free energy of the medium is a function of temperature and stretch ratios of the elastic elements

$$f = f(\theta, \ldots, \lambda_1^i, \lambda_2^i, \lambda_3^i, \ldots),$$

where θ is the temperature, and λ_1^i, λ_2^i, and λ_3^i are the stretch ratios of the ith elastic element.

Statement 8. The deviator of the Cauchy stress tensor of the ith elastic element is calculated by the formulas of the nonlinear elasticity theory

$$\operatorname{dev} T_i = \operatorname{dev}\Big(\rho \sum_{k=1}^{3} \lambda_k^i \, \frac{\partial f}{\partial \lambda_k^i}\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i\Big), \qquad \operatorname{dev}(\,\cdot\,) = (\,\cdot\,) - \frac{1}{3}\operatorname{tr}(\,\cdot\,), \tag{9}$$

where ρ is the mass density of the medium, and $\boldsymbol{n}_1^i$, $\boldsymbol{n}_2^i$, and $\boldsymbol{n}_3^i$ form an orthogonal triple of eigenvectors of the stretch tensor $V_i = \lambda_1^i\, \boldsymbol{n}_1^i \otimes \boldsymbol{n}_1^i + \lambda_2^i\, \boldsymbol{n}_2^i \otimes \boldsymbol{n}_2^i + \lambda_3^i\, \boldsymbol{n}_3^i \otimes \boldsymbol{n}_3^i$ of the ith elastic element.

In the example considered (see Fig. 1), the mass density of the free energy f is a function of the temperature θ and the stretch ratios of the first (λ_1^1, λ_2^1, and λ_3^1), fifth (λ_1^5, λ_2^5, and λ_3^5), and sixth (λ_1^6, λ_2^6, and λ_3^6) elastic elements. Let us recall that the notions of deformation gradients for individual elements of the scheme are not used here; therefore, we cannot argue that the stretch tensors V_1, V_5, and V_6 appear as a result of polar decomposition of the corresponding deformation gradients. It follows from the formulas given below that the stretch tensors V_1, V_5, and V_6 are symmetric and indifferent. The mass density of the free energy f is used to calculate the Cauchy stress tensors of the elastic elements T_1, T_5, and T_6.

In further formulas, we use the dots above the symbols to indicate the derivatives of these quantities with respect to time under the condition that the change in the considered quantities is traced for a fixed particle of the medium. The dot near the closing bracket has the same meaning of the derivative of the expression in brackets with respect to time (for a fixed particle of the medium). To finalize the description of the behavior of the elastic elements, we have to define how the stretch tensors behave with time. We propose the following statement.

Statement 9. For the ith elastic element, the material derivative of the stretch tensor $\dot{V}_i$ with respect to time is calculated by the formula

$$\frac{2}{\delta_i}\, Y_i^{0.5} D_i Y_i^{0.5} = \dot{Y}_i - Y_i W_R^{\mathrm{t}} - W_R Y_i, \qquad W_R = \dot{R} R^{\mathrm{t}}, \tag{10}$$

where $Y_i = V_i^{2/\delta_i}$ ($\delta_i > 0$) and R is the rotation tensor in the polar decomposition $F = VR$ of the deformation gradient of the medium F into the left stretch tensor V and rotation R.

Corollary 1. The known formulas of the theory of nonlinear medium elasticity, which describe the time variation of the stretch ratios of the ith elastic element,

$$\dot{\lambda}_k^i = \lambda_k^i\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot D_i, \qquad k = 1, 2, 3, \tag{11}$$

and the rate of work in this element

$$T_i \cdot D_i = \rho \sum_{k=1}^{3} \frac{\partial f}{\partial \lambda_k^i}\, \dot{\lambda}_k^i \tag{12}$$

are the consequences of Eq. (10) if the parameter δ_i is a constant.

Let us prove equality (11). For this purpose, we write Eq. (10) in a more convenient (for analysis) form

$$\frac{2}{\delta_i}\, D_i = Y_i^{-0.5} \dot{Y}_i Y_i^{-0.5} - Y_i^{0.5} W_R^{\mathrm{t}} Y_i^{-0.5} - Y_i^{-0.5} W_R Y_i^{0.5}. \tag{13}$$

The left and right sides of Eq. (13) are scalarly multiplied by the expression $\lambda_k^i\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i$:

$$\lambda_k^i\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot \frac{2}{\delta_i}\, D_i = \lambda_k^i\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot \Big(Y_i^{-0.5} \dot{Y}_i Y_i^{-0.5} - Y_i^{0.5} W_R^{\mathrm{t}} Y_i^{-0.5} - Y_i^{-0.5} W_R Y_i^{0.5}\Big). \tag{14}$$

The first term in the right side of Eq. (14) is written with the use of eigenvectors and eigenvalues of the stretch tensor of the ith elastic element:

$$\lambda_k^i\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot Y_i^{-0.5} \dot{Y}_i Y_i^{-0.5} = \lambda_k^i \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot V_i^{-1/\delta_i} (V_i^{2/\delta_i})^{\boldsymbol{\cdot}}\, V_i^{-1/\delta_i}$$

$$= \lambda_k^i \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot \Big(\sum_{n=1}^{3} (\lambda_n^i)^{-1/\delta_i} \boldsymbol{n}_n^i \otimes \boldsymbol{n}_n^i\Big)\Big(\sum_{j=1}^{3} (\lambda_j^i)^{2/\delta_i} \boldsymbol{n}_j^i \otimes \boldsymbol{n}_j^i\Big)^{\boldsymbol{\cdot}}\Big(\sum_{m=1}^{3} (\lambda_m^i)^{-1/\delta_i} \boldsymbol{n}_m^i \otimes \boldsymbol{n}_m^i\Big). \tag{15}$$

With the use of the identity

$$\boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot (\boldsymbol{n}_j^i \otimes \boldsymbol{n}_j^i)^{\boldsymbol{\cdot}} \equiv 0$$

and the rule of scalar multiplication of second-rank tensors

$$A \cdot BC = CA^{\mathrm{t}} \cdot B^{\mathrm{t}},$$

which is valid for arbitrary tensors A, B, and C, we finally obtain the expression

$$\lambda_k^i\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot Y_i^{-0.5} \dot{Y}_i Y_i^{-0.5} = (\lambda_k^i)^{-2/\delta_i+1}\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot \sum_{j=1}^{3} \frac{2}{\delta_i}\, (\lambda_j^i)^{2/\delta_i-1} \dot{\lambda}_j^i\, \boldsymbol{n}_j^i \otimes \boldsymbol{n}_j^i = \frac{2}{\delta_i}\, \dot{\lambda}_k^i. \tag{16}$$

We can easily see that the second and third terms in the right side of Eq. (14) are equal to zero. Let us illustrate it by an example of the second term, which can be simplified by using the rule of scalar multiplication of second-rank tensors:

$$-\lambda_k^i\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot Y_i^{0.5} W_R^{\mathrm{t}} Y_i^{-0.5}$$

$$= -\lambda_k^i\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot \Big(\sum_{n=1}^{3} (\lambda_n^i)^{1/\delta_i}\, \boldsymbol{n}_n^i \otimes \boldsymbol{n}_n^i \Big) W_R^{\mathrm{t}} \Big(\sum_{m=1}^{3} (\lambda_m^i)^{-1/\delta_i}\, \boldsymbol{n}_m^i \otimes \boldsymbol{n}_m^i \Big) = -\lambda_k^i\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot W_R^{\mathrm{t}}.$$

As the scalar product of symmetric and antisymmetric tensors is equal to zero, we finally obtain

$$-\lambda_k^i\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot Y_i^{0.5} W_R^{\mathrm{t}} Y_i^{-0.5} = 0. \tag{17}$$

Thus, equality (14) with allowance for Eqs. (16) and (17) acquires the form

$$\lambda_k^i\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot \frac{2}{\delta_i}\, D_i = \frac{2}{\delta_i}\, \dot{\lambda}_k^i,$$

whence there follows the validity of equality (11).

Expression (12) is proved on the basis of Eq. (11) with the use of the equality

$$\rho \sum_{k=1}^{3} \frac{\partial f}{\partial \lambda_k^i}\, \dot{\lambda}_k^i = \rho \sum_{k=1}^{3} \frac{\partial f}{\partial \lambda_k^i}\, \lambda_k^i\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot D_i.$$

Condition (9) means that the Cauchy stress tensor of the ith elastic element is calculated by the formula

$$T_i = p_i I + \rho \sum_{k=1}^{3} \lambda_k^i\, \frac{\partial f}{\partial \lambda_k^i}\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i,$$

where p_i is an indefinite parameter. The trace of the deformation rate tensor of the elastic element is equal to zero: $\operatorname{tr} D_i = I \cdot D_i = 0$. The above-presented reasoning allows us to conclude that Eq. (12) is valid.

Corollary 2. In the general case, where the parameters δ_i can change their values in the course of material deformation, the rate of time variation of the stretch ratio of the ith elastic element and the rate of work in this element are found by the formulas

$$\begin{gathered} \dot{\lambda}_k^i = \lambda_k^i\, \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot D_i + \frac{\dot{\delta}_i}{\delta_i}\, \lambda_k^i\, \ln(\lambda_k^i), \\ T_i \cdot D_i = \rho \sum_{k=1}^{3} \frac{\partial f}{\partial \lambda_k^i}\, \dot{\lambda}_k^i - \frac{\dot{\delta}_i \rho}{\delta_i} \sum_{k=1}^{3} \frac{\partial f}{\partial \lambda_k^i}\, \lambda_k^i \ln(\lambda_k^i). \end{gathered} \tag{18}$$

It seems reasonable to use the changes in the parameters δ_i with time to model the growth of damages in the material. If the values of δ_i remain unchanged, then the properties of the elastic elements are determined by the known equations of the theory of nonlinear elastic media (11) and (12). If the material becomes damaged, the ratio between the macroscopic deformations and structural deformations of the elements is changed, which is taken into account by the parameters δ_i. Equality (18) testifies that some part of work performed in the elastic element is spent on changing the free energy, whereas the other part of work is spent on increasing the medium damage. In the present paper, we consider incompressible materials only; therefore, we assume that the emergence of damages should not lead to changes in the medium volume, for instance, in the case of breakdown of aggregates of carbon black particles in rubber during its deformation. Elastic fibers composed of an oriented polymer are formed between the parts of the aggregates rather than pores. The medium remains incompressible.

3. Incompressibility of the Elastic Elements. Let us demonstrate that the condition of incompressibility of the elastic elements in the scheme of the mechanical behavior of the medium does not contradict the condition of satisfaction of equality (10). For this purpose, the left and right sides of equality (13) are scalarly multiplied by the unit tensor $I = \boldsymbol{n}_1^i \otimes \boldsymbol{n}_1^i + \boldsymbol{n}_2^i \otimes \boldsymbol{n}_2^i + \boldsymbol{n}_3^i \otimes \boldsymbol{n}_3^i$. As a result, we obtain

$$I \cdot \frac{2}{\delta_i} D_i = \sum_{k=1}^{3} \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot \Big(Y_i^{-0.5} \dot{Y}_i Y_i^{-0.5} - Y_i^{0.5} W_R^{\mathrm{t}} Y_i^{-0.5} - Y_i^{-0.5} W_R Y_i^{0.5} \Big).$$

The left side of this equality is equal to zero, because the trace of the deformation rate tensor is equal to zero. Repeating transformations similar to those performed in obtaining Eqs. (14)–(16), we find

$$0 = \sum_{k=1}^{3} (\lambda_k^i)^{-2/\delta_i} \boldsymbol{n}_k^i \otimes \boldsymbol{n}_k^i \cdot \sum_{j=1}^{3} \Big(\frac{2}{\delta_i} (\lambda_j^i)^{2/\delta_i - 1} \dot{\lambda}_j^i - \frac{2(\lambda_j^i)^{2/\delta_i}}{\delta_i^2} \dot{\delta}_i \ln \lambda_j^i \Big) \boldsymbol{n}_j^i \otimes \boldsymbol{n}_j^i$$

$$= \frac{2}{\delta_i} \sum_{j=1}^{3} \frac{\dot{\lambda}_j^i}{\lambda_j^i} - \frac{2\dot{\delta}_i}{{\delta_i}^2} \sum_{j=1}^{3} \ln \lambda_j^i = \frac{2}{\delta_i} \Big(\ln (\lambda_1^i \lambda_2^i \lambda_3^i) \Big)^{\boldsymbol{\cdot}} - \frac{2\dot{\delta}_i}{{\delta_i}^2} \ln (\lambda_1^i \lambda_2^i \lambda_3^i).$$

The solution of this equation is the product $\lambda_1^i \lambda_2^i \lambda_3^i = 1$, which was to be proved.

4. Mechanical Properties of the Viscous and Plastic Elements. The following statements are used to take into account the properties of viscosity and plasticity.

Statement 10. The deviator of the Cauchy stress tensor of the jth viscous element is calculated by the formulas of the theory of nonlinear viscous fluid

$$\operatorname{dev} T_j = 2\eta_j D_j, \tag{19}$$

where the shear viscosity coefficient is a non-negative function of state parameters $\eta_j \geqslant 0$.

Statement 11. The deviator of the Cauchy stress tensor of the plastic element is calculated by the formulas of the plastic flow theory

$$D_n = \sqrt{\frac{D_n \cdot D_n}{\operatorname{dev} T_n \cdot \operatorname{dev} T_n}} \operatorname{dev} T_n, \tag{20}$$

where n is the number of the plastic element.

For modeling the plastic flow process, it is necessary to exclude the ambiguity in Eq. (20). A mathematical expression that relates the deformation rate tensor of the plastic element to the deformation rate tensor of the medium can be used for this purpose.

Statement 12. The specific features of the mechanical behavior of the plastic element are determined by a proportional dependence between the intensity of the deformation rate tensor of the plastic element and the intensity of the deformation rate tensor of the medium proper:

$$\sqrt{D_n \cdot D_n} = \varkappa_n \sqrt{D \cdot D}\,.$$

Here the factor $\varkappa_n$ is a non-negative function described by the dependence

$$\varkappa_n = \begin{cases} 0, & \Phi_n(V, \ldots) < g_n, \\ \zeta(g_n), & \Phi_n(V, \ldots) = g_n. \end{cases}$$

The fluidity function Φ_n, which is used to formulate the criterion of evolution of plastic deformations in the medium, is a function of the stretch tensor V and other state parameters of the medium. Plastic deformation of the medium occurs if and only if the fluidity function Φ_n has the maximum value during the entire history of medium existence:

$$g_n = \max \Phi_n(V, \ldots).$$

5. Transmission Elements in the Model. In this paper, we consider isotropic materials; therefore, we propose to use simple relations between the stress tensors and deformation rate tensors at the left and right points of the transmission element.

Statement 13. The transmission element in the model serves to increase the deformation rate tensor by a factor of ν_k with a corresponding decrease in the Cauchy stress tensor:

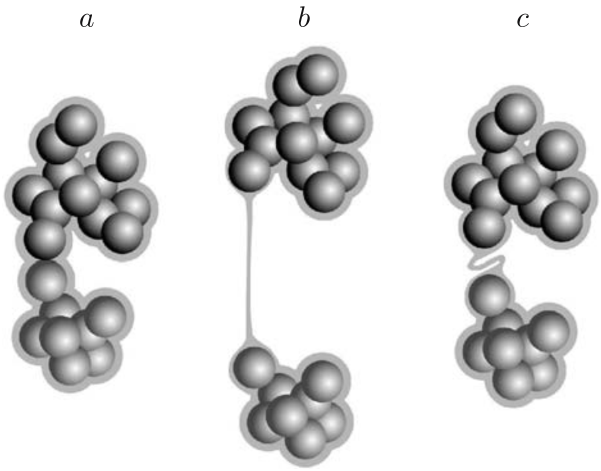

Fig. 2

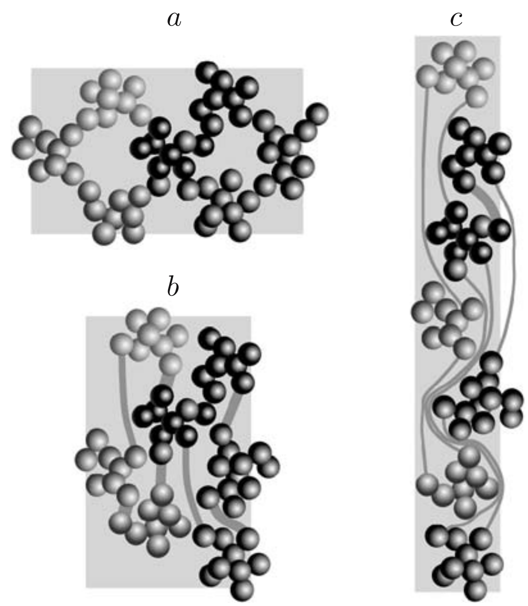

Fig. 3

Fig. 2. Illustration to the hypothesis of formation of oriented polymer fibers in the gap between the neighboring carbon black aggregates during rubber deformation: (a) locations of the neighboring aggregates at the initial time; (b) formation of oriented fibers owing to slipping of the polymer chains from the layers into the gap between the aggregates under stretching; (c) whipping of the fibers after load removal.

Fig. 3. Locations of aggregates of carbon black particles at large deformations of rubber: (a) initial locations of the aggregates of the filler particles; (b) locations of the aggregates with twofold elongation of the material specimen; (c) locations of the aggregates with threefold elongation of the material specimen.

$$T_k^l = \nu_k T_k^r, \qquad D_k^l = \frac{1}{\nu_k}\, D_k^r$$

(ν_k is a non-negative function of the state parameters of the medium and k is the number of the transmission element).

In the model illustrated in Fig. 1, the transmission elements serve to increase the deformation rate tensors of the points C and D over the deformation rate tensor of the point A:

$$D_A = \frac{1}{\nu_2}\, D_C, \qquad D_A = \frac{1}{\nu_3}\, D_D.$$

The condition of the absence of sources and sinks of energy in the transmission elements (4) is satisfied automatically.

Transmission elements are used in the model for a more accurate description of the processes in real materials. On one hand, the use of these elements allows a more accurate description of the mechanical behavior of the material; one the other hand, it becomes possible to understand and quantify the role of the processes at the structural level. Let us illustrate it by an example of rubber, a nanocomposite capable of elongating by more than a factor of 5 under stretching and returning to a state close to the initial state after the loading is removed. Rubbers are elastomers filled by carbon black. Carbon black particles have radii between 10 and 20 nm and are united into aggregates touching each other and forming a rigid skeleton in the elastomer. During deformation, apparently, the polymer chains slip off from the polymer layers near the filler particles into the gaps between the aggregates, where high-strength fibers of a uniaxially oriented polymer are formed (Fig. 2). As a result, the macroscopic strength of elastomers increases by an order of magnitude (as compared with the material without the filler), with simultaneous growth of deformations at the moment of material rupture.

It should be noted that rubbers are unusual nanocomposites. Methods of self-consistency or methods of studying the properties on periodic cells are inapplicable to these materials. Large deformations of the material

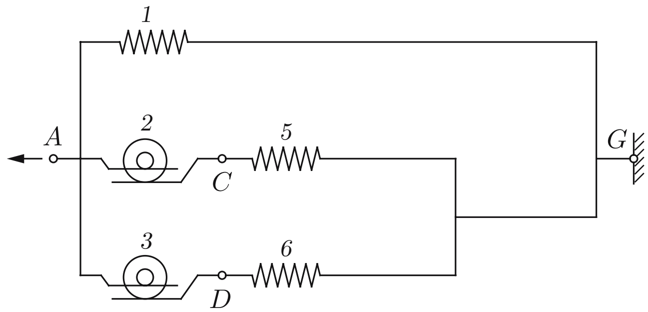

Fig. 4. Scheme shown in Fig. 1 under deformation conditions with both the plastic element and both viscous elements being out of operation (notation the same as in Fig. 1).

lead to significant changes in its structure (Fig. 3). The condition of medium incompressibility can only be fulfilled if the closely located aggregates of carbon black particles move away from each other in the case of twofold elongation of the material sample and diverge to extremely large distances in the case of fourfold elongation. The length of the fibers connecting the aggregates should increase by a factor of several tens thereby (see Fig. 3). The area of the hatched region remains unchanged. The images of the aggregates of carbon black particles can be arranged in this region only if the length of the connecting fibers can be substantially increased. Transmission elements in the model allow us to take into account this behavior of the oriented polymer fibers.

Special attention should be paid to the fact that the equation of the evolution of the left tensor of stretching of the elastic element V_i is chosen in the form (10). Let us consider the model shown in Fig. 1 in the case where the viscous and plastic elements are out of operation (Fig. 4), for instance, in the second cycle of material deformation along a given trajectory with an extremely high rate. All possible plastic deformations already occurred in the first cycle. The second cycle proceeds within a time period much shorter that the characteristic time of the viscoelastic process. It can be naturally expected that the medium would behave as a hyperelastic material under these conditions. Such a situation occurs if the following conditions are used in the model: $\delta_1 = 1$, $\nu_2 = \delta_5$, and $\nu_3 = \delta_6$. Indeed, in accordance with Eq. (10), the behavior of the elastic element 1 in Fig. 4 is consistent with the known laws of the nonlinear elasticity theory:

$$2Y^{0.5} D Y^{0.5} = \dot{Y} - Y W_R^{\mathrm{t}} - W_R Y, \qquad Y = V_1^2. \tag{21}$$

The left stretch tensor of the material V satisfies the same equation:

$$2VDV = (V^2)^{\boldsymbol{\cdot}} - V^2 W_R^{\mathrm{t}} - W_R V^2.$$

The evolution of the tensors V and V_1 is uniquely determined by the deformation rate tensor D and the spin W_R of the medium, which coincide with the unit tensor in the non-deformed material. These facts allow us to speak about the equality of V and V_1.

Let us study the behavior of the elastic elements 5 and 6. The following dependences are valid for the transmission elements:

$$D = D_5/\nu_2, \qquad D = D_6/\nu_3.$$

This means that the equations of the evolution of the stretch tensors have the form (21), but the tensor Y in these equations is related to V_5 and V_6 by the dependences

$$Y = {V_5}^{2/\nu_2}, \qquad Y = {V_6}^{2/\nu_3}.$$

The tensors V_5 and V_6 in the non-deformed material coincide with the unit tensor. This allows us to conclude that there exists a functional dependence of the stretch tensors of the elastic elements on the left stretch tensor of the material:

$$V_1 = V, \qquad V_5 = V^{\nu_2}, \qquad V_6 = V^{\nu_3}.$$

In the case considered, the viscous and plastic elements are out of operation. Hence, we have a hyperelastic behavior of the medium whose free energy is uniquely determined by the left stretch tensor of the material. To obtain this result, we have to choose appropriate values for the parameters δ_1, δ_5, and δ_6.

6. Verification of the Dissipation Inequality. The constitutive equations of the model should yield the dissipation inequalities

$$T \cdot D - \rho(\dot{f} + s\dot{\theta}) - \frac{\boldsymbol{h} \cdot \operatorname{grad} \theta}{\theta} \geqslant 0, \tag{22}$$

where $\boldsymbol{h}$ is the heat flux and s is the entropy of the material. The temperature gradient is determined in the actual configuration.

Statement 14. The material entropy s and the heat flux $\boldsymbol{h}$ are calculated by the formulas of nonequilibrium thermodynamics

$$s = -\frac{\partial f}{\partial \theta}, \qquad \boldsymbol{h} = -c_h \operatorname{grad} \theta, \tag{23}$$

where $c_h > 0$ is the thermal conductivity.

Statement 15. The following inequality should be valid for all elastic elements:

$$\dot{\delta}_i \Big(\sum_{k=1}^{3} \frac{\partial f}{\partial \lambda_k^i} \lambda_k^i \ln(\lambda_k^i) \Big) \leqslant 0. \tag{24}$$

Let us prove the validity of inequality (22) for a material whose mechanical behavior is determined by the scheme shown in Fig. 1. The proof for any other material is similar. Let us decompose the scalar product $T \cdot D$ using Eqs. (1), (2), (5), and (6) and applying some obvious transformations:

$$T \cdot D = (T_1 + T_2^l + T_3^l) \cdot D = T_1 \cdot D + T_2^r \cdot D_C + T_3^r \cdot D_D$$

$$= T_1 \cdot (D_A - D_B) + T_1 \cdot D_B + T_2^r \cdot (D_C - D_F) + T_2^r \cdot D_F + T_3^r \cdot (D_D - D_E) + T_3^r \cdot D_E$$

$$= T_1 \cdot D_1 + T_1 \cdot D_B + T_2^r \cdot D_5 + T_2^r \cdot D_F + T_3^r \cdot D_6 + T_3^r \cdot D_E.$$

After further simplification with the use of Eqs. (1), (3), (7), and (8), we obtain

$$T \cdot D = T_1 \cdot D_1 + T_1 \cdot D_B + T_5 \cdot D_5 + T_5 \cdot D_F + T_6 \cdot D_6 + T_7 \cdot D_E$$

$$= T_1 \cdot D_1 + T_4 \cdot D_B + T_5 \cdot D_5 + T_5 \cdot D_F + T_6 \cdot D_6 + T_7 \cdot (D_E - D_F) + T_7 \cdot D_F$$

$$= T_1 \cdot D_1 + T_4 \cdot D_4 + T_5 \cdot D_5 + T_5 \cdot D_8 + T_6 \cdot D_6 + T_7 \cdot D_7 + T_7 \cdot D_8 = T_1 \cdot D_1 + \sum_{k=4}^{8} T_k \cdot D_k.$$

Thus, the power of stresses acting in the medium is equal to the sum of the powers of stresses in each element of the scheme of the mechanical behavior of the medium, except for the transmission elements. As a result, the dissipation inequality (22) acquires the form

$$T_1 \cdot D_1 + \sum_{k=4}^{8} T_k \cdot D_k - \rho(\dot{f} + s\dot{\theta}) - \frac{\boldsymbol{h} \cdot \operatorname{grad} \theta}{\theta} \geqslant 0. \tag{25}$$

In the example considered, the density of the free energy of the material is a function of temperature and stretch ratios of the elastic elements:

$$f = f(\theta, \lambda_1^1, \lambda_2^1, \lambda_3^1, \lambda_1^5, \lambda_2^5, \lambda_3^5, \lambda_1^6, \lambda_2^6, \lambda_3^6).$$

Taking into account Eqs. (18) and (23), we substitute into inequality (25) the value of the material derivative of the density of the free energy

$$\dot{f} = \frac{\partial f}{\partial \theta} \dot{\theta} + \sum_{k=1,5,6} \sum_{i=1}^{3} \frac{\partial f}{\partial \lambda_i^k} \dot{\lambda}_i^k.$$

As a result, we obtain the constraint

$$\sum_{n=4,7,8} T_n \cdot D_n - \sum_{i=1,5,6} \sum_{k=1}^{3} \frac{\dot{\delta}_i \rho}{\delta_i} \frac{\partial f}{\partial \lambda_k^i} \lambda_k^i \ln(\lambda_k^i) + \frac{c_h \operatorname{grad}\theta \cdot \operatorname{grad}\theta}{\theta} \geqslant 0.$$

Using the properties of the viscous element (19) and plastic element (20), we transform this constraint to

$$\sum_{n=4,7} T_n \cdot \frac{\operatorname{dev} T_n}{2\eta_n} + T_8 \cdot \sqrt{\frac{D_8 \cdot D_8}{\operatorname{dev} T_8 \cdot \operatorname{dev} T_8}} \operatorname{dev} T_8 - \sum_{i=1,5,6} \sum_{k=1}^{3} \frac{\dot{\delta}_i \rho}{\delta_i} \frac{\partial f}{\partial \lambda_k^i} \lambda_k^i \ln(\lambda_k^i) + \frac{c_h \operatorname{grad}\theta \cdot \operatorname{grad}\theta}{\theta} \geqslant 0.$$

Using the identity

$$A \cdot \operatorname{dev} A = \operatorname{dev} A \cdot \operatorname{dev} A,$$

we write the dissipation inequality in the final form:

$$\sum_{n=4,7} \frac{1}{2\eta_n} \operatorname{dev} T_n \cdot \operatorname{dev} T_n + \sqrt{\frac{D_8 \cdot D_8}{\operatorname{dev} T_8 \cdot \operatorname{dev} T_8}} \operatorname{dev} T_8 \cdot \operatorname{dev} T_8$$

$$- \sum_{i=1,5,6} \sum_{k=1}^{3} \frac{\dot{\delta}_i \rho}{\delta_i} \Big(\sum_{k=1}^{3} \frac{\partial f}{\partial \lambda_k^i} \lambda_k^i \ln(\lambda_k^i) \Big) + \frac{c_h \operatorname{grad}\theta \cdot \operatorname{grad}\theta}{\theta} \geqslant 0.$$

As the shear viscosities η_n, the parameters δ_i, and the thermal conductivity c_h are positive, the dissipation inequality is valid for all elastic elements under constraints (24). Inequalities (24) have the following physical meaning: the change in the parameters δ_i should occur only in one direction determined by Eqs. (24). The damage of the material can be only increasing.

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